

Nonextensive Effects on Chandrasekhar's Dynamical Friction

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ABSTRACT

The motion of a point like object of mass M passing through the background potential of massive collisionless particles ($m \ll M$) suffers a steady deceleration named dynamic friction. In his classical work, Chandrasekhar assumed a Maxwellian velocity distribution in the halo and neglected the self gravity of the wake induced by the gravitational focusing of the mass M . In this paper, by relaxing the validity of the Maxwellian distribution due to the presence of long range forces, we derive an analytical formula for the dynamic friction in the context of the q -nonextensive kinetic theory. In the extensive limiting case ($q = 1$), the classical Gaussian Chandrasekhar result is recovered. As an application, the dynamic friction timescale for Globular Clusters spiraling to the galactic center is explicitly obtained. Our results suggest that the problem concerning the large timescale as derived by numerical N -body simulations or semi-analytical models can be understood as a departure from the standard extensive Maxwellian regime as measured by the Tsallis nonextensive q -parameter.

Subject headings: Dynamical Friction, Nonextensivity, Globular Clusters

1. Introduction

It is widely known that a massive object of mass M such as a Globular Cluster passing through a background of non-colliding particles suffers a gravitational effect usually referred to as Dynamical Friction (DF). Historically, this problem was first studied by Chandrasekhar (1943) who first recognized its true dissipative nature. He analyzed the idealized case where a point mass moves through an infinity, homogeneous sea of field particles and showed that a fraction of the kinetic energy of the incoming object is transferred to the stellar collisionless population whose distribution was described by a Maxwellian velocity.

The DF mechanism is now a classical effect for description and evolution of almost all many-body astrophysical systems. Some examples involve the formation of stellar galactic nuclei via merging of old Globular Clusters (GCs)

(Tremaine *et al.* 1975), the transformation from non-nucleated dwarf galaxies into nucleated ones (Oh & Lin 2000), the behavior of radio galaxies in galaxy clusters (Nath 2008), nonlinear gaseous medium (Kin & Kin 2009) and the field particles with a mass spectrum (Ciotti 2010). Traditionally, such investigations were carried out in the framework of Newtonian gravity, however, alternative gravity theories like the Modified Newtonian Dynamics (MOND) has also been considered (Nipoti *et al.* 2008).

In the last few years, several authors have discussed the problem related to the dynamical friction timescale (t_{df}) of a GC orbiting dwarf galaxies or infalling satellite galaxies in clusters (Read *et al.* 2006; Goerdt *et al.* 2006; Sánchez-Salsedo *et al.* 2006; Nath 2008; Cowsik *et al.* 2009; Inoue 2009; Namouni 2010; Gan *et al.* 2010). In particular, when dwarf galaxies have a cored dark matter halo with constant density distribution in its center it has been found that the DF effects will be considerably modified. Some analyses based on N -body simulations (Goerdt *et al.* 2006; Inoue 2009) and semi-analytical models have shown that the ex-

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pected sinking of CGs to the galactic center may take a time beyond the age of the universe.

In order to explain this suppression effect and the consequent problem of extremely long t_{df} , some authors introduced additional ingredients, among them: the loss mass of the GCs by stripping and heating tidal (Gan *et al.* 2010; Boylan-Kolchin *et al.* 2008), and the interaction between the dark matter halo and the GC (Oh & Lin 2000; Read *et al.* 2006; Inoue 2009). All these possibilities are based on the standard Chandrasekhar's DF formula derived by assuming a Maxwellian distribution. Here we investigate a different route by considering that the solution for this problem is related to a proper extension of the underlying statistical approach.

It is well known that the so-called nonextensive statistical approach provides an analytical extension of Boltzmann-Gibbs (BG) statistical mechanics which is very convenient when long-range forces are present and/or the system is out (but close) to a thermal equilibrium state. This ensemble theory is based on the formulation of a generalized entropy proposed by Tsallis (1988, 2009)

$$S_q = k_B \frac{1 - \sum_{i=1}^W p_i^q}{1 - q}, \quad (1)$$

which reduces in the limit $q \rightarrow 1$ to the BG entropy $S_{BG} = -k_B \sum_{i=1}^W p_i \ln p_i$, since p_i is the probability of finding the systems in the microstate i , W is the number of microstates and k_B is the Boltzmann constant. However, when the index $q \neq 1$, the entropy of the system is nonextensive, i.e., given two subsystems A and B , the entropy is no more additive in the sense that $S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B)$. The long-range interactions are associated to the last term on the r.h.s. which accounts for correlations between the subsystems with the index q quantifying the degree of statistical correlations. Such a statistical description has been successfully applied to many complex physical systems ranging from physics to astrophysics and plasma physics, among them: the electrostatic plane-wave propagation in a collisionless thermal plasma (Lima, Silva & Santos 2000), the peculiar velocity function of galaxies clusters (Lavagno *et al.* 1998), gravothermal instability (Taruya & Sakagami 2002), the kinetic concept of Jeans gravitational instability (Lima, Silva & Santos 2002), and the radial and projected

density profiles for two large classes of isothermal stellar systems (Lima & de Souza 2005). A wide range of physical applications can also be seen in Gell-Mann & Tsallis 2004 (see also <http://tsallis.cat.cbpf.br/biblio.htm> for an updated bibliography).

In this letter, by assuming that a self-gravitating collisionless gas is described by the nonextensive kinetic theory (Silva *et al.* 1998; Lima *et al.* 2001), we derive a new analytical formula for dynamical friction which generalizes the Chandrasekhar result. As an application, the DF timescale (t_{df}) for GCs falling in the galaxies center is derived for the case of a singular isothermal sphere. This result suggest that the long timescales for GCs can be understood as a departure from the extensive regime. In other words, there is no suppression of the DF since the long time can be just the statistical price to pay by the presence of long range forces acting on the gravitational systems.

2. Dynamical Friction and Nonextensive Effects

By following Chandrasekhar (1943), the DF deceleration on a test mass M moving with velocity v_M in a homogeneous and isotropic distribution of identical field particles of mass m and number density n_0 reads:

$$\frac{d\mathbf{v}_M}{dt} = -16\pi^2 (\ln \Lambda) G^2 M m \frac{\int_0^{v_M} f(v) v^2 dv}{v_M^3} \mathbf{v}_M, \quad (2)$$

where G is the gravitational constant, m is the mean mass of field stars and $f(v)$ represents their velocity distribution. The parameter $\Lambda = p_{max}/p_{min}$ depends on the ratio of the the maximum (p_{max}) and minimum (p_{min}) impact parameters of the encounters contributing to generate the dragging force.

In the applications of DF, it is usually assumed that the distribution function of the stellar velocity field can be described by a Maxwellian distribution (Binney & Tremaine 2008; Fellhauer 2008)

$$f(X_\star) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} e^{-X_\star^2}, \quad (3)$$

where $X_\star = v/\sqrt{2}\sigma$ denotes a normalized velocity with σ indicating their dispersion. The integration

of (2) results:

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi \ln \Lambda G^2 M \rho(r)}{v_M^3} H_1(X_M) \mathbf{v}_M, \quad (4)$$

where $\rho(r) = n_0 m$ and the function $H_1(X_M)$ is given by

$$H_1(X_M) = \text{erf}(X_M) - \frac{2X_M}{\sqrt{\pi}} e^{-X_M^2}, \quad (5)$$

with $\text{erf}(X_M)$ defining the error function as

$$\text{erf}(X_M) = \frac{2}{\sqrt{\pi}} \int_0^{X_M} e^{-X_*^2} dX_*. \quad (6)$$

Now, in order to investigate the nonextensive effects on the Chandrasekhar theory, let us consider that the stellar field obeys the following power-law (Silva, Plastino & Lima 1998, Lima, Silva & Plastino 2001, Lima & de Souza 2005):

$$f(X_*) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} A_q e_q(X_*) \quad (7)$$

where the so-called q -exponential is defined by

$$e_q(X_*) = [1 - (1 - q)X_*^2]^{\frac{1}{1-q}}, \quad (8)$$

and the quantity A_q denotes a normalization constant which depends on the interval of the q -parameter. For values of $q < 1$, the positiveness of power argument means that distribution above exhibits a cut-off in the maximal allowed velocities. In this case, all velocities lie on the interval $(0, v_{max})$ and their maximum value is $v_{max} = \sqrt{2}\sigma/\sqrt{1-q}$. Taking this into account one may show that the normalization constant A_q can be written in terms of Gamma functions as follows:

$$\begin{aligned} A_q &= (1-q)^{1/2} \left(\frac{5-3q}{2}\right) \left(\frac{3-q}{2}\right) \frac{\Gamma(\frac{1}{1-q} + \frac{1}{2})}{\Gamma(\frac{1}{1-q})}, \quad q < 1 \\ A_q &= (q-1)^{3/2} \frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{1}{q-1} - \frac{3}{2})}, \quad q > 1 \end{aligned} \quad (9)$$

For generic values of $q \neq 1$, the DF (7) is a power law, whereas for $q = 1$ it reduces to the standard Maxwell-Boltzmann distribution function (3) since $A_1 \rightarrow 1$ at this limit. Formally, this result follows directly from the known identity, $\lim_{d \rightarrow 0} (1 + dy)^{\frac{1}{d}} = \exp(y)$ (Abramowitz & Stegun 1972). The distribution

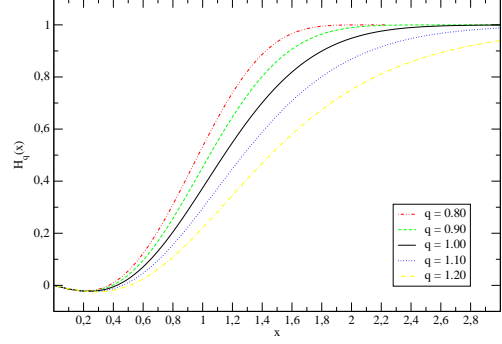


Fig. 1.— The $H_q(X)$ function. The solid black curve is the result based on Chandrasekhar theory ($H_1(x)$). The remaining curves show the q -corrections for several values of the q -index.

(7) is uniquely determined from two simple requirements (Silva *et al.* 1998): (i) isotropy of the velocity space, and (ii) a suitable nonextensive generalization of the Maxwell factorizability condition, or equivalently, the assumption that $f(v) \neq f(v_x)f(v_y)f(v_z)$. The kinetic foundations of the above distribution were also investigated in a deeper level through the generalized Boltzmann's equation, in particular, it was also shown that the kinetic version of the Tsallis entropy satisfies an extended H_q -theorem (Lima, Silva & Plastino 2001).

Now, by considering that the power-law distribution (7) is a valid description for the stellar velocity distribution we conclude that the expression describing the DF in this extended framework takes the following form:

$$\frac{d\mathbf{v}_M}{dt} = -\frac{16\pi^2 G^2 (\ln \Lambda) M \rho(r)}{v_M^3} A_q \times \int_0^{X_M} X_*^2 e_q(X_*) dX_* \mathbf{v}_M, \quad (10)$$

which after an elementary integration can be rewritten as:

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi G^2 \ln \Lambda \rho(r) M}{v_M^3} \left(\frac{2}{5-3q}\right) H_q(X_M) \mathbf{v}_M, \quad (11)$$

where $H_q(X_M)$ is the general function depending on the q -parameter (compare with Eq. (5))

$$H_q(X_M) = I_q(X_M) - \frac{2X_M}{\sqrt{\pi}} A_q e_q^{2-q}(X_M). \quad (12)$$

In the above expression, the integral

$$I_q(X_M) = \frac{2A_q}{\sqrt{\pi}} \int_0^{X_M} e_q(X_\star) dX_\star, \quad (13)$$

is the q -generalization of the error function (see Eq.(6)).

As one may check, the nonextensive expression for the DF (including the auxiliary functions H_q and I_q) reduces to the Chandrasekhar result in the Gaussian limit ($q \rightarrow 1$). It shows clearly that the collective effect from gravitational interactions of M (with all stars of the field) is strongly dependent on the statistical model. An interesting aspect of the above formulae is that the results are expressed by analytical expressions. In principle, they can be useful for semi-analytical implementations because the easy comparison with the standard approach (see next section). Naturally, we are also advocating here that the idealized framework based on the Maxwellian distribution (Chandrasekhar 1943) may be in the root of some theoretical difficulties shown by N -body simulations, like the ones related to the decay orbits of GCs.

3. Decay of Globular Orbits

In order to illustrate some consequences of the above derivation, let us now analyze the nonextensive solution for the decaying orbit of a GC in the stellar galactic field. As a GC orbits through the galaxy field, it is subject to DF due to its interaction with the stellar distribution. By assuming spherically symmetric star distribution, the dragging force decelerates the cluster motion which loses energy thereby spiraling toward the galaxy center. Therefore, whether the GC is initially on a circular orbit of radius r_i , it is convenient to define an average DF timescale, t_{df} , as the time required for the cluster reach the galaxy center. For the sake of simplicity, we also consider that the mass density distribution of the galaxy is described by the singular isothermal sphere

$$\rho(r) = \frac{1}{4\pi G} \left(\frac{v_c}{r} \right)^2, \quad (14)$$

with v_c being circular speed and $\sigma = v_c/\sqrt{2}$ the velocity dispersion. This simplified mass distribution has the benefit of having a planar rotation curve and therefore might be considered as a crude

but minimally realistic distribution for the external region of normal galaxies. Following standard lines, the frictional force undergone by the cluster of mass M moving with speed v_c through the stellar field now reads:

$$F = - \left(\frac{2}{5-3q} \right) G \ln \Lambda \left(\frac{M}{r} \right)^2 H_q(1), \quad (15)$$

where $H_q(1)$ is the general function (12) written in the coordinate $X = (v_c/\sigma\sqrt{2}) = 1$. Note also that the integral $I_q(X_M)$ defined in (13) now reduces to

$$I_q(1) = \frac{2A_q}{\sqrt{\pi}} {}_2F_1 \left(\frac{1}{q-1}, \frac{1}{2}; \frac{3}{2}; 1-q \right), \quad (16)$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Either from the above representation or from the integral form (13), we see that the error function $erf(1)$ is obtained as a particular case in the extensive regime, that is, $I_1(1) = erf(1) \approx 0.8427$ (Abramowitz & Stegun 1972). It means that $H_1(1) = erf(1) - (2/\sqrt{\pi})e^{-1} \approx 0.428$ (Binney & Tremaine 2008).

Now, returning to expression (15), we recall that the dragging force is tangential to the cluster orbits, and, therefore, the cluster gradually loses angular momentum per unit mass L at a rate $dL/dt = Fr/M$. Since $L = rv_c$ we can rewritten equation (15) as

$$r \frac{dr}{dt} = - \left(\frac{2}{5-3q} \right) \left(\frac{GM}{v_c} \right) \ln \Lambda H_q(1). \quad (17)$$

By solving this differential equation subjected to the initial condition, $r(0) = r_i$, we find that the cluster reaches the galaxy center after a time

$$t_{df}^{(q)} = \left(\frac{5-3q}{2} \right) \frac{0.5v_cr_i^2}{GM \ln \Lambda H_q(1)}. \quad (18)$$

This nonextensive timescale for decaying orbits of GCs generalizes the Chandrasekhar result (see Binney & Tremaine 2008) which is readily recovered in the Gaussian extensive limit ($q = 1$).

At this point, it is interesting to compare the above nonextensive prediction with the standard result based on the Chandrasekhar approach. To begin with, let us assume typical values for the parameters r_i , v_c and M , namely: $r_i = 2Kpc$, $v_c = 250kms^{-1}$ and $M = 10^6 M_\odot$. With these

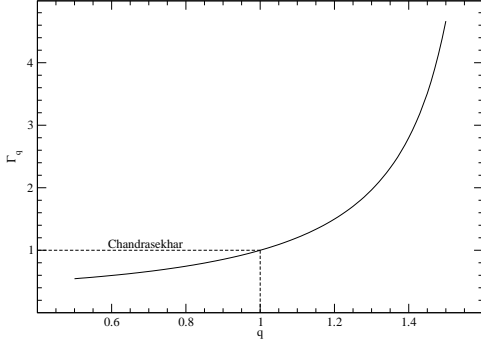


Fig. 2.— Behavior of the relative time scale ratio Γ_q . We see that for $q > 1$, the characteristic nonextensive time scale for dynamic friction can be much greater than in the standard Chandrasekhar approach.

choices we get:

$$t_{df}^{(q)} \approx \frac{1.14 \times 10^{11}}{H_q(1) \ln \Lambda} \left(\frac{5-3q}{2} \right) \left(\frac{r_i}{2 \text{ kpc}} \right)^2 \times \left(\frac{v_c}{250 \text{ km s}^{-1}} \right) \left(\frac{10^6 M_\odot}{M} \right) \text{ yr}, \quad (19)$$

which reduces to the standard value in limiting case ($q = 1$) as given by Binney & Tremaine (2008). The nonextensive corrections are more directly quantified by introducing the dynamic time ratio, $\Gamma(q) \equiv t_{df}^{(q)} / t_{df}^{(1)}$, where $t_{df}^{(1)}$ denotes the Chandrasekhar result. By using (18) we find

$$\Gamma(q) = \left(\frac{5-3q}{2} \right) \frac{H_1(1)}{H_q(1)} = \left(\frac{5-3q}{2} \right) \frac{0.428}{H_q(1)}. \quad (20)$$

where the function $H_q(X)$ was defined by Eq. (11).

In Figure 1, we display the nonextensive corrections for a large range of the nonextensive q -parameter. As a general result, we see that the $\Gamma(q)$ ratio is strongly dependent on the q -parameter. The nonextensive time scale is greater or less than the extensive Chandrasekhar result depending on the interval of the q -parameter. Note also that $t_{df}^{(q)}$ is greater or smaller than $t_{df}^{(1)}$ if $q > 1$ or $q < 1$, respectively.

4. Conclusions

We have derived the q -dynamic friction force for a point mass moving through a homogeneous

background in the context of the nonextensive kinetic theory. Simple and analytical forms were obtained, and, as should be expected, they smoothly reduce to the standard Chandrasekhar results in the extensive limiting case ($q = 1$). However, for $q \neq 1$ a large variety of qualitatively different behaviors are predicted when the free parameter q is continuously varied (see Figs. 1 e 2). As an application, we have discussed the dynamical timescale for a globular cluster collapsing to the center of a massive dark matter halo described by an isothermal sphere. The results presented here suggest that the problem related to the large timescale shown by numerical N -body simulations and semi-analytical models may naturally be solved (with no ad hoc mechanism) by taking a proper q -nonextensive distribution with parameter greater than unity. Applications to more realistic density profiles like the lowered nonextensive halos distribution (Silva, de Souza & Lima 2009; Cardone, Leubner & Del Popolo) and a detailed comparison with semi-analytical models will be discussed in a forthcoming communication.

Acknowledgments: JMS is supported by FAPESP Agency and JASL by FAPESP and CNPq (Brazilian Research Agencies).

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